One more variant of discrete gravity having "naive" continuum limit

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Some variant of discrete quantum theory of gravity having "naive" continuum limit is constructed. It is shown that in a highly compressed state of universe a sort of "high-temperature expansion" is valid and, thus, the confinement of "color" takes place at early stage of universe expansion. In the considered theory any nontrivial representation of the local Lorentz group (i.e. spinor, vector and so on fields) play the role of color. The arguments are given in favor of a significant noncompact packing of quantized field modes in momentum space.

I. INTRODUCTION

Since the traditional methods of quantization of gravity in four dimensions prove to be inconsistent because of ultraviolet divergences, a natural idea has arisen that qualitatively different physics takes place at supersmall distances (of the order of the Planck length and smaller).

At present, a predominant opinion among theoretical physicists is that superstring theory is a fundamental physical theory. In a ten-dimensional space, superstring theory is self-consistent. The superstring theory involves gravitational interaction.

However, one encounters an extremely difficult problem within the string ideology, the problem of the compactification of six dimensions and the construction of a long-wavelength physics in four dimensions. Therefore, recently, actual progress in solving many problems of the quantum theory of gravity and quantum cosmology along this line has not been made.

The aforesaid justifies the existence of certain other ideas underlying the fundamental quantum field theory and, first of all, the theory of gravity. In our opinion, the most interesting idea is the idea of discrete space-time, which is the main subject of the present paper [27].

The idea of the discreteness of space-time (as applied to the theory of gravity) was first formulated in the pioneering work by Regge [1] long before the appearance of string theory. According to Regge, the role of smooth Riemannian spaces is played by piecewise flat spaces, namely, simplicial complexes (necessary information from the theory of simplicial complexes is given at the beginning of the next section). To each one-dimensional simplex (edge) is assigned its length, so that, if the set

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of three edges forms a boundary of a two-dimensional simplex (a triangle), then the lengths of these edges satisfy the triangle inequalities. Thus, the geometry of a complex is completely defined. The quantity representing an analogue of the Riemann tensor on a smooth Riemannian space proves to be nonzero only on a set of $(D-2)$-dimensional simplices (D is the dimension of the simplicial complex); i.e., the curvature tensor becomes a distributions.

The detailed account of the Regge calculus is given in [2]. An approach to discrete geometry similar to the Regge calculus can also be found in [3],[4].

Despite the obvious elegance of the Regge calculus, this theory proves to be very inconvenient when passing to quantum theory. Indeed, the independent variables that determine Regge action are the lengths of one-dimensional simplices subject to a large number of constraints, namely, to triangle inequalities. Moreover, the introduction of Dirac fields to the theory creates a new difficulty consisting in the absence of orthonormal bases in the explicit form. Possibly, this is the reason why the variant of discrete gravity based on the so-called B-F-theory is currently being developed more intensively.

The B-F-theory is developed on the basis of the action for gravity theory in the Palatini form. I write out the action for a four-dimensional gravity theory with a massless Dirac field, especially because the introducing notations are used below

$$I = \int \varepsilon_{abcd} \left\{ -\frac{1}{l_P^2} R^{ab} \wedge e^c \wedge e^d + \frac{1}{6} \Theta^a \wedge e^b \wedge e^c \wedge e^d \right\}, \tag{1.1}$$

$$\mathrm{d}\,\omega^{ab} + \omega_c^a \wedge \omega^{cb} = \frac{1}{2} R^{ab},$$

$$\Theta^a = \frac{i}{2} \left(\overline{\psi}\,\gamma^a\,\mathcal{D}_\mu\psi - \mathcal{D}_\mu\overline{\psi}\,\gamma^a\,\psi \right)\,\mathrm{d}\,x^\mu. \tag{1.2}$$

Here,

$$\omega^{ab} = \omega_\mu^{ab}\,\mathrm{d}\,x^\mu \tag{1.3}$$

is a connection 1-form in a certain orthonormal basis $\{e_a^\mu\}$, $e^a = e_\mu^a\,\mathrm{d}\,x^\mu$, $g_{\mu\nu} = e_\mu^a e_\nu^a$ is a metric tensor, l_P is the Planck length, ψ is a Dirac field, γ^a are the Dirac matrices, and $\mathcal{D}_\mu\psi$ is the covariant derivative of the Dirac field.

The most characteristic property of the B-F-theory consists in that the curvature tensor 2-form is equal to zero. However, while the B-F-theory really describes gravity in a three-dimensional space, this is not so in higher dimensional spaces. For example, in a four-dimensional space with the action (1.1), the curvature tensor is not equal to zero. But it is not the only difficulty of the theory: the introduction of the matter is also awkward.

The detailed description of discrete quantum theory of gravity based on the B-F formalism can be found in [5]-[14].

In contrast to the multidimensional case, considerable computational progress has been made in two-dimensional discrete quantum gravity (see [15], [16]). We refer the reader to [17] for the relation between the three-dimensional quantum Yang-Mills theory on a lattice and three-dimensional gravity.

In the present paper, we propose a new version of discrete quantum theory of gravity. This new theory differs both from the Regge theory and from the discrete variant of the B-F-theory. Just as in the B-F-theory, the connection in our theory is represented by the elements a gauge group. However, unlike the B-F-theory, all fundamental variables in the theory proposed here are defined directly on the elements of the simplicial complex itself. In particular, the gauge group G elements that play the role of connection 1-forms are defined on one-dimensional simplices. In contrast to the B-F-theory, we explicitly introduce an analogue of a tetrad 1-form in our theory. The elements of a tetrad 1-form are also defined on one-dimensional simplices and belong to the real vector space with the basis formed by Dirac matrices. The presence of a tetrad form in the theory allows us to introduce a Dirac field whose elements are defined at the vertices of a simplicial complex and they are transformed by a spinor representation of the group G. Using these fields, one can easily construct a lattice action which is invariant relative to the gauge transformations. It is important that this action obviously manifests the continuum limit, at least at "naive" level. In the naive continuum limit, this action is reduced to a standard action of the continuum theory of gravity (1.1). Further the quantization of discrete gravity is performed. This means the determination of the fundamental partition function, which represents a functional gauge-invariant integral over the introduced fields. It is shown qualitatively that this quantum theory displays the tendency to degenerate into macroscopic continuum theory. It turns out that the correct determination of the partition function requires that the gauge group should be compact, which is equivalent to a metric with Euclidean signature. Euclidean signature in fundamental quantum gravity has been introduced much earlier in the work [18].

It seems very interesting that in highly compressed state of universe (which is possible at small times near the singularity) a sort of "high-temperature expansion" is valid. Therefore, the confinement of "color" takes place at early stage of universe expansion. Note that in the considered theory any nontrivial representation of the local Lorentz or gauge group G (i.e. spinor, vector and so on fields) plays the role of color.

In conclusion, the arguments are given in favour of the dynamics of discrete quantum gravity leads to the significant loosening of packing of field modes in momentum space. Thus, the packing of modes in momentum space in the corresponding continuum theory turns out to be significantly noncompact. From here it follows that the universe has the weight smaller by many orders in comparison with the case of the usual quantum field theory.

Here we use the results obtained in the previous publications [19], [20].

It is necessary mention here two interesting works [21], [22], in which the formulations of discrete quantum gravity are the most close to the one presented here. But the constructions in [21], [22] are based on the hypercubical lattices but not the simplicial complexes.

II. DISCRETE QUANTUM GRAVITY. DEFINITION OF ACTION

Let $\mathfrak{K}$ be a 4-dimensional simplicial complex admitting geometrical realization. The definition and required properties of simplicial complexes can be found in [19]. A detailed theory of simplicial complexes is given, for example, in [23] – [24]. Below instead of "simplicial complex" we say simply "complex", and the concepts in the following pairs are treated as synonyms: 0-simplex and vertex; 1-simplex and edge; 2-simplex and triangle; 3-simplex and tetrahedron. The finite complexes with a 4-disk topology are interesting here. Such complexes have a boundary $\partial \mathfrak{K}$ which is 3-dimensional complex with topology of 3-sphere S^3. Denote by $\alpha_q, \ q = 0, 1, 2, 3, 4$ the number of q-simplexes of the complex $\mathfrak{K}$. The indexes $i, j, k, l, \ \ldots$ run through the complex vertices: $a_i, \ a_j$ and so on. Two vertices are called adjacent if these two vertices are the boundary vertices of the same edge.

For convenience I give here the definition of orientation of simplexes and complexes.

A simplex

$$s^r = \varepsilon(a_0, \ a_1, \ \ldots, \ a_r) \equiv \varepsilon \, a_0 \, a_1 \, \ldots \, a_r \qquad (2.1)$$

has an orientation, or is oriented, if every order of its vertices is assigned a sign "+" or "-", so that orders differing by an odd permutation correspond to opposite signs. Thus if $\varepsilon = 1$ the orientation of simplex (2.1) is given by the orders $(a_0, \ a_1, \ \ldots, \ a_r)$ or $-(a_1, \ a_0, \ \ldots, \ a_r)$. Let $(a_0, \ \ldots, \ a_{i-1}, \ a_{i+1}, \ \ldots, \ a_r)$ be the face of a simplex s^r obtained by eliminating one vertex a_i from the sequence $a_0, \ a_1, \ \ldots, \ a_r$. By definition, the orientation of this face, given by

$$B_i^{r-1} = (-1)^i \, \varepsilon(a_0, \ \ldots, \ a_{i-1}, \ a_{i+1}, \ \ldots, \ a_r), \qquad (2.2)$$

is called an iduced orientation of the simplex s^r.

Denote by D the maximum value of number r in (2.1) for all simplexes of complex. In the considered case $D = 4$. Thus D is the dimension of complex. Two oriented D-dimensional simplices s_1^D and s_2^D of a D-dimensional simplicial complex are called concordantly oriented if either the simplices s_1^D and s_2^D have no common $(D-1)$-dimensional faces or the orientation of their common $(D-1)$-dimensional face B^{D-1} induced by the orientation of the simplex s_1^D is opposite to the orientation of the same face B^{D-1} induced by the orientation of the simplex s_2^D. A D-dimensional simplicial complex $\mathfrak{K}$ is called orientable if there exists such an orientation for all its D-dimensional simplices that any pair of its D-dimensional simplices is concordantly oriented. The concordant orientation of D-dimensional simplices defines the orientation of the complex, and namely this orientation of D-simplices is regarded as positive.

Evidently, interesting for us the complex $\mathfrak{K}$ is orientable.

Below index A enumerates 4-simplices. Introduce the following notation for oriented 1-simplices in the case when the vertexes a_i and a_j belong to the 4-simplex with index A

$$X_{ij}^A = a_i a_j = -X_{ji}^A . \tag{2.3}$$

Let

$$s_A^4 = a_{i_0} a_{i_1} a_{i_2} a_{i_3} a_{i_4} \tag{2.4}$$

be an positively oriented 4-simlex with index A. An oriented frame of a simplex (2.4) at a vertex a_{i_0} is the ordered set of four oriented 1-simplices (2.3) such that an even permutation of these 1-simplices does not change the orientation while an odd permutation changes the orientation of the frame to the opposite. By definition, the frame

$$\mathcal{R}^{A\,i_0} = \left(X_{i_0 i_1}^A,\ X_{i_0 i_2}^A,\ X_{i_0 i_3}^A,\ X_{i_0 i_4}^A \right) \tag{2.5}$$

is oriented positively.

Let γ^a, $a, b, c, \ldots = 1, 2, 3, 4$ be 4×4 Dirac matrices with Euclidean signature. Thus all Dirac matrices as well as matrix

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 , \qquad \mathrm{tr}\ \gamma^5 \gamma^a \gamma^b \gamma^c \gamma^d = 4\,\varepsilon^{abcd} \tag{2.6}$$

are Hermitian. To each vertex a_i, we assign the Dirac spinors ψ_i and $\overline{\psi}_i$ each of whose components assumes values in a complex Grassman algebra. In the case of Euclidean signature, the spinors ψ_i and $\overline{\psi}_i$ are independent variables and are

interchanged under the Hermitian conjugation. The Dirac matrixes act from the left to the spinors ψ_i and from the right to the spinors $\overline{\psi}_i$.

Let us assign to each oriented edge $a_i a_j$ an element of the group $Spin(4)$

$$\Omega_{ij} = \Omega_{ji}^{-1} = \exp\left(\frac{1}{2}\omega_{ij}^{ab}\sigma^{ab}\right), \quad \sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \tag{2.7}$$

Holonomy element Ω_{ij} of the gravitational field executes a parallel transformation of spinor ψ_j from vertex a_j of edge $a_i a_j$ to neighboring vertex a_i. We denote by V a linear space with basis γ^a. Let each oriented edge $a_i a_j$ be put in correspondence with element $\hat{e}_{ij} \equiv e_{ij}^a \gamma^a \in V$, such that

$$\hat{e}_{ij} = -\Omega_{ij}\hat{e}_{ji}\Omega_{ij}^{-1}. \tag{2.8}$$

The notations $\overline{\psi}_{Ai}$, ψ_{Ai}, $\hat{e}_{Aij}$, Ω_{Aij} and so on indicate that edge $X_{ij}^A = a_i a_j$ belongs to 4-simplex with index A. Here, the sign " $-$ " in (2.8) is due to the fact that e_{Aij} and e_{Aji} are the values of the 1-form on the edges $X_{ij}^A = a_i a_j$ and $X_{ji}^A = a_j a_i = -a_i a_j = -X_{ij}^A$ (which are oriented mutually oppositely), respectively. The "facing" from the elements of a holonomy group on the right-hand side of Eq. (2.8) are necessary since the element e_{Aji} must be paralled-translated from the vertex a_{Aj} to the vertex a_{Ai} to compare this element with the element e_{Aij}. The quantities assigned to each oriented edge $a_i a_j$ and satisfying to Eq. (2.8) are called 1-forms.

By assumption, the complex $\mathfrak{K}$ has a disk topology. For such a complex, the concept of orientation can be introduced. We define the orientation of the complex by defining the orientation of each 4-simplex. In this case, if two 4-simplices have a common tetrahedron, the two orientations of the tetrahedron, which are defined by the orientations of these two 4-simplices, are opposite. In our case, the complex obviously has only two orientations.

Let a_{Ai}, a_{Aj}, a_{Ak}, a_{Al}, and a_{Am} be all five vertices of a 4-simplex with index A and $\varepsilon_{Aijklm} = \pm 1$ depending on whether the order of vertices $a_{Ai}\, a_{Aj}\, a_{Ak}\, a_{Al}\, a_{Am}$ defines the positive or negative orientation of this 4-simplex. In addition, $\varepsilon_{ijklm} = 0$ if at least two indices coincide. We can now write the Euclidean action in the model in question

$$I = \frac{1}{5 \times 24}\sum_{A}\sum_{i,j,k,l,m}\varepsilon_{Aijklm}\,\mathrm{tr}\,\gamma^5 \times$$

$$\times\left\{-\frac{1}{2\,l_P^2}\Omega_{Ami}\Omega_{Aij}\Omega_{Ajm}\hat{e}_{Amk}\hat{e}_{Aml}+\right.$$

$$\left.+\frac{1}{24}\hat{\Theta}_{Ami}\hat{e}_{Amj}\hat{e}_{Amk}\hat{e}_{Aml}\right\}, \tag{2.9}$$

$$\hat{\Theta}_{Aij} = \frac{i}{2}\gamma^a \left(\overline{\psi}_{Ai}\gamma^a \Omega_{Aij}\psi_{Aj} - \overline{\psi}_{Aj}\Omega_{Aji}\gamma^a\psi_{Ai}\right) \equiv$$

$$\equiv \Theta^a_{Aij}\gamma^a \in V. \tag{2.10}$$

The quantity $\hat{\Theta}_{Aij}$, as well as the whole action (2.9), represents an Hermitean operator. One can easily verify that 1-form (2.10), just as the 1-form $\hat{e}_{ij}$, satisfies relation (2.8). This fact is established by the repeated application of the formula

$$S^{-1}\gamma^a S = S^a_b \gamma^b, \tag{2.11}$$

where

$$S \equiv \exp\frac{1}{2}\varepsilon_{ab}\sigma^{ab}, \qquad \varepsilon_{ab} = -\varepsilon_{ba} = \varepsilon^a_b,$$

$$S^a_b \equiv \left(\exp\varepsilon\right)^a_b = \delta^a_b + \varepsilon^a_b + \frac{1}{2}\varepsilon^a_c\varepsilon^c_b + \dots. \tag{2.12}$$

The volume of a 4-complex is given by

$$V_A = \frac{1}{4!} \times \frac{1}{5!} \times$$

$$\times \sum_A \sum_{i,j,k,l,m} \varepsilon_{Aijklm}\varepsilon^{abcd} e^a_{Ami}e^b_{Amj}e^c_{Amk}e^d_{Aml}. \tag{2.13}$$

Here, factor $1/4!$ is required since the volume of a four-dimensional parallelepiped with generatrices e^a_{Ami}, e^b_{Amj}, e^c_{Amk}, and e^d_{Aml} is $4!$ times larger than the volume of a 4-simplex with the same generatrices, while factor $1/5!$ is due to the fact that all five vertices of each simplex are taken into account independently.

The dynamic variables are quantities Ω_{ij} and $\hat{e}_{ij}$, which describe the gravitational degrees of freedom, and fields $\overline{\psi}_i$ and ψ_i, which are material fermion fields (other material fields are not considered here).

In the space of fields, there acts a gauge group according to the following rule. To each vertex a_{Ai}, let us assign an element of the group $S_{Ai} \in \mathrm{Spin}(4)$. According to the principle of gauge invariance, the fields Ω, e, ψ, and the transformed fields

$$\tilde{\Omega}_{Aij} = S_{Ai}\Omega_{Aij}S^{-1}_{Aj},$$

$$\tilde{e}_{Aij} = S_{Ai}e_{Aij}S^{-1}_{Ai},$$

$$\tilde{\psi}_{Ai} = S_{Ai}\psi_{Ai}, \qquad \tilde{\overline{\psi}}_{Ai} = \overline{\psi}_{Ai}S^{-1}_{Ai} \tag{2.14}$$

are physucally equivalent. This means that the action (2.9) is invariant under the transformations (2.14). Under the gauge transformatios (2.14), the 1-form Θ is transformed in the same way as the form e

$$\tilde{\hat{\Theta}}_{Aij} = S_{Ai}\hat{\Theta}_{Aij}S^{-1}_{Ai}. \tag{2.15}$$

The last formula is verified with the help of Eqs. (2.11), (2.12) and (2.14). Gauge invariance of the action (2.9) and of the volume (2.13) is established by using Eqs. (2.14) and (2.15).

It is natural to interpret the quantity

$$l_{ij}^2 \equiv \frac{1}{4} \, \mathrm{tr} \, (\hat{e}_{ij})^2 = \sum_{a=1}^{4} (e_{ij}^a)^2 \tag{2.16}$$

as the square of the length of the edge $a_i a_j$. Thus, the geometric properties of a simplicial complex prove to be completely defined.

Now, let us show that, in the limit of slowly varying fields, the action (2.9) reduces to the continuum action of gravity, minimally connected with with a Dirac field, in a four-dimensional Euclidean space.

Consider a certain subset of vertices from the simplicial complex and assign the coordinates (real numbers)

$$x_{Ai}^\mu \equiv x^\mu(a_{Ai}), \qquad \mu = 1, \, 2, \, 3, \, 4 \tag{2.17}$$

to each vertex a_{Ai} from this subset. We stress that these coordinates are defined only by their vertices rather than by the higher dimension simplices to whom these vertices belong; moreover, the correspondence between the vertices from the subset considered and the coordinates (2.17) is one-to-one.

Suppose that

$$| \, x_{Ai}^\mu - x_{Aj}^\mu \, | \sim a \,, \tag{2.18}$$

where the parameter a is of the order of the lattice spacing. Estimates (2.18) has the sense if the complex contains a very large number of simplices and its geometric realization is an almost smooth four-dimensional surface [28]. Suppose also that the four 4-vectors

$$\mathrm{d} \, x_{Aji}^\mu \equiv x_{Ai}^\mu - x_{Aj}^\mu, \qquad i \neq j, \quad i = 1, \, 2, \, 3, \, 4 \tag{2.19}$$

are linearly independent and

$$\begin{vmatrix} \mathrm{d} \, x_{Am1}^1 & \mathrm{d} \, x_{Am1}^2 & \cdots & \mathrm{d} \, x_{Am1}^4 \\ \cdots & \cdots & \cdots \cdots \\ \mathrm{d} \, x_{Am4}^1 & \mathrm{d} \, x_{Am4}^2 & \cdots & \mathrm{d} \, x_{Am4}^4 \end{vmatrix} > 0 \,, \tag{2.20}$$

provided that the frame $\left(X_{m1}^A, \, \ldots, \, X_{m4}^A \right)$ is positively oriented. Inequality (2.20) implies that positively oriented local coordinates are introduced on the almost flat

surface considered. Here, the differentials of coordinates (2.19) correspond to one-dimensional simplices $a_{Aj}a_{Ai}$, so that, if the vertex a_{Aj} has coordinates x^μ_{Aj}, then the vertex a_{Ai} has the coordinates $x^\mu_{Aj} + \mathrm{d}\, x^\mu_{Aji}$.

In the continuum limit, the holonomy group elements (2.7) are close to the identity element, so that the quantities ω^{ab}_{ij} tend to zero being of the order of $O(\mathrm{d}\, x^\mu)$. Thus one can consider the following system of equation for $\omega_{Am\mu}$

$$\omega_{Am\mu}\, \mathrm{d}\, x^\mu_{Ami} = \omega_{Ami}, \quad i = 1,\, 2,\, 3,\, 4\,. \tag{2.21}$$

In this system of linear equation, the indices A and m are fixed, the summation is carried out over the index μ, and index runs over all its values. Since the determinant (2.20) is positive, the quantities $\omega_{Am\mu}$ are defined uniquely. Suppose that a one-dimensional simplex X^A_{mi} belong to four-dimensional simplices with indices $A_1,\, A_2,\, \ldots\,,\, A_r$. Introduce the quantity

$$\omega_\mu \left[\frac{1}{2}\left(x_{Am} + x_{Ai} \right) \right] \equiv \frac{1}{r} \left\{ \omega_{A_1 m\mu} +\, \ldots\, + \omega_{A_r m\mu} \right\}, \tag{2.22}$$

which is assumed to be related to the midpoint of the segment $[x^\mu_{Am},\, x^\mu_{Ai}]$. Recall that the coordinates x^μ_{Ai} just as the differentials (2.19), depend only on vertices but not on the higher dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities

$$\omega_{A_1 mi} = \omega_{A_2 mi} =\, \ldots\, = \omega_{A_r mi}\,. \tag{2.23}$$

It follows from (2.19) and (2.21)–(2.23) that

$$\omega_\mu \left(x_{Am} + \frac{1}{2}\, \mathrm{d}\, x_{Ami} \right) \mathrm{d}\, x^\mu_{Ami} = \omega_{Ami}\,. \tag{2.24}$$

The value of the field ω_μ in (2.24) on each one-dimensional simplex is uniquely defined by this simplex.

Next, we assume that the fields ω_μ smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to $O\big((\mathrm{d}\, x)^2\big)$ inclusive

$$\Omega_{Ami}\, \Omega_{Aij}\, \Omega_{Ajm} = \exp\left[\frac{1}{2}\, \mathfrak{R}_{Am\mu\nu}\, \mathrm{d}\, x^\mu_{Ami}\, \mathrm{d}\, x^\nu_{Amj} \right], \tag{2.25}$$

where

$$\mathfrak{R}_{Am\mu\nu} = \partial_\mu \omega_{Am\nu} - \partial_\nu \omega_{Am\mu} + [\omega_{Am\mu},\, \omega_{Am\nu}]\,. \tag{2.26}$$

On the right-hand side of (2.25), as well as in equality (2.26), all fields are taken at the vertex a_{Am} of a four-dimensional simplex A as is indicated by the subscript Am. When deriving formula (2.25), we used the Hausdorff formula.

In exact analogy with (2.21), let us write out the following relations for a tetrad field without explanations

$$e_{Am\mu}\, \mathrm{d}\, x^{\mu}_{A\,mi} = e_{A\,mi}\,. \tag{2.27}$$

Using (2.7) and (2.21), we can rewrite the 1-form (2.10) as

$$\Theta_{Aij} = \gamma^a\, \frac{i}{2}\, \big[\, \overline{\psi}_{Ai}\, \gamma^a\, \mathcal{D}_\mu\, \psi_{Ai} - \overline{\mathcal{D}_\mu\, \psi}_{Ai}\, \gamma^a\, \psi_{Ai}\,\big]\, \mathrm{d}\, x^{\mu}_{A\,ij}\,, \tag{2.28}$$

to within $O(\mathrm{d}\, x)$; here,

$$\mathcal{D}_\mu\, \psi_{Ai} = \partial_\mu\, \psi_{Ai} + \omega_{Ai\mu}\, \psi_{Ai}\,. \tag{2.29}$$

Before rewriting the action (2.9) in the continuum limit, we give the following obvious formula

$$\sum_{\sigma(Am)} \varepsilon_{\sigma(Am)}\, \mathrm{d}\, x^{\mu}_{A\,mi}\, \mathrm{d}\, x^{\nu}_{A\,mj}\, \mathrm{d}\, x^{\lambda}_{A\,mk}\, \mathrm{d}\, x^{\rho}_{A\,ml} =$$

$$= 24\, \varepsilon^{\mu\nu\lambda\rho}\, v_{SA}\,. \tag{2.30}$$

Here, $\varepsilon^{\mu\nu\lambda\rho}$ is a completely antisymmetric symbol, which is equal to unity when $(\mu\,\nu\,\lambda\,\rho) = (1\,2\,3\,4)$ (compare with (2.20)), and v_{SA} is the volume of the geometric realization of simplex A in a four-dimensional Euclidean space when the Euclidean coordinates of the geometric realization of the simplex are equal to the corresponding coordinates of its vertices (2.17). The factor 24 in (2.30) is necessary since the volume v_{SA} of the four-dimensional simplex on the right-hand side is less than the volume of a four-dimensional parallelepiped constructed on the vectors $\mathrm{d}\, x^{\mu}_{A\,mi},\, \dots,\, \mathrm{d}\, x^{\mu}_{A\,ml}$ by a factor of 24.

Applying formulas (2.25)–(2.30), changing the summation to integration and taking into account that

$$\mathfrak{R} \equiv \frac{1}{2}\, \sigma^{ab}\, R^{ab}_{\mu\nu}\, \mathrm{d}\, x^{\mu} \wedge \mathrm{d}\, x^{\nu}\,, \quad e = \gamma^a\, e^a_\mu\, \mathrm{d}\, x^{\mu}\,,$$

$$\Theta = \gamma^a\, \frac{i}{2}\, \big[\, \overline{\psi}\gamma^a\, \mathcal{D}_\mu\, \psi - \overline{\mathcal{D}_\mu\psi}\, \gamma^a\, \psi\,\big]\, \mathrm{d}\, x^{\mu}\,, \tag{2.31}$$

we obtain the well known expression (1.1) for the action (2.9) in the continuum limit:

Thus, in the naive continuum limit, the action (2.9) proves to be equal to the action of gravity with a Λ-term and a metric with Euclidean signature that is minimally coupled to a Dirac field.

III. QUANTIZATION OF DISCRETE GRAVITY

Let us determine the partition function Z for a discrete Euclidean gravity [29]. Let us enumerate the zero-dimensional (vertices) and one-dimensional (edges) simplices by indices $\mathcal{V}$ and $\mathcal{E}$, respectively, and denote by $\psi_\mathcal{V}$, $\Omega_\mathcal{E}$, etc. the corresponding variables. By definition,

$$Z = \text{const} \cdot \left(\prod_\mathcal{E} \int d\,\Omega_\mathcal{E} \int d\,e_\mathcal{E} \right) \times$$
$$\times \left(\prod_\mathcal{V} d\,\overline{\psi}_\mathcal{V}\, d\,\psi_\mathcal{V} \right) \exp\left(-I \right) . \tag{3.1}$$

Here, const is a certain normalizing factor, $d\,\Omega_\mathcal{E}$ is the Haar measure on the group $\text{Spin}(4)$,

$$d\,e_\mathcal{E} \equiv \prod_a d\,e_\mathcal{E}^a , \qquad e_\mathcal{E} = e_\mathcal{E}^a\, \gamma^a , \tag{3.2}$$

and

$$d\,\overline{\psi}_\mathcal{V}\, d\,\psi_\mathcal{V} \equiv \prod_\nu d\,\overline{\psi}_{\mathcal{V}\nu}\, d\,\psi_{\mathcal{V}\nu} . \tag{3.3}$$

The index ν in (3.3) enumerates individual components of the spinors $\psi_\mathcal{V}$ and $\overline{\psi}_\mathcal{V}$, such that we have a product of the differentials of all independent generators of the Grassman algebra of Dirac spinors in (3.3). The action I in (3.1) is defined by formula (2.9).

Note that the measure (3.2) is determined correctly in view of invariance of the Haar measure and the relations (2.7) and (2.8). Therefore, one can really assume that the measure (3.2) is related to the set of edges.

Obviously, all the measures used in the functional integral (3.1) are invariant under the gauge transformations (2.14). Since the action I (2.9) in (3.1) is also gauge invariant, the partition function (3.1) is invariant under the action of the gauge group (2.14).

Consider the partition function (3.1) with a zero Λ-term in the absence of fermions. In this case, the integral over the 1-form $e_\mathcal{E}$ becomes Gaussian

$$Y\{\Omega\} = \int D z \cdot \exp\left(\frac{1}{2}\, z_m\, \mathcal{M}_{mn}\, z_n \right) . \tag{3.4}$$

Here, $\{z_m\}$, $m = 1, \ldots, Q$ denotes a set of real variables $\{\omega_\mathcal{E}^a\}$ and $\mathcal{M}_{mn}$ is a real

symmetrical matrix depending on the elements of the holonomy group $\Omega_{\mathcal{E}}$. Thus,

$$\frac{1}{2} z_m \, \mathcal{M}_{mn} \, z_n \equiv \frac{1}{l_P^2} \frac{1}{5} \cdot \frac{1}{24} \sum_{A,\, m} \sum_{\sigma(Am)} \varepsilon_{\sigma(Am)} \times$$

$$\times \operatorname{tr} \left(\gamma^5 \, \Omega_{A \, mi} \, \Omega_{A \, ij} \, \Omega_{A \, jm} \, e_{A \, mk} \, e_{A \, ml} \right). \tag{3.5}$$

Denote by $\{\lambda_q\}$, where $q = 1, \dots, Q$, a set of eigenvalues of the matrix $\mathcal{M}_{mn}$. Let $\varepsilon_q = \operatorname{sign} \lambda_q$. Since, in general, there are both negative and positive eigenvalues among $\{\lambda_q\}$, the integral (3.4) should be redefined. This is done by passing to Lorentzian signature. Under this procedure, the eigenvalues are transformed by the rule

$$\lambda_q \to e^{i\varphi} \, \lambda_q \,,$$

where $\varphi = 0$ in the Euclidean space and $\varphi = \pi/2$ in the case of the Minkowski signature. Thus, the Euclidean Gaussian integral

$$\mathcal{I}_E = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\, z \cdot \exp\left(\frac{1}{2} \lambda \, z^2 \right) \tag{3.6}$$

reduces to the Fresnel integral in the Minkowski signature

$$\mathcal{I}_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\, z \cdot \exp\left(\frac{i}{2} \lambda \, z^2 \right) = \sqrt{\frac{i}{\lambda}} =$$

$$= (i)^{\frac{\varepsilon}{2}} \frac{1}{\sqrt{|\lambda|}} \,, \tag{3.7}$$

where $\varepsilon = \operatorname{sign} \lambda$. Let us perform the analytic continuation

$$\lambda \to e^{-i\varphi} \, \lambda$$

on the right-hand side of Eq. (3.7) and set $\varphi = \pi/2$. Thus, we recover the Euclidean signature of a metric and obtain the following value for integral (3.6)

$$\mathcal{I}_E = (i)^{\frac{\varepsilon+1}{2}} \frac{1}{\sqrt{|\lambda|}} \,. \tag{3.8}$$

Now, using Eq. (3.8), we redefine the integral (3.4) of interest

$$Y\{\Omega\} = \operatorname{const} \prod_q (i)^{\frac{\varepsilon_q + 1}{2}} \, |\lambda_q|^{-1/2} \,. \tag{3.9}$$

If there are fermion fields in the theory, one should first calculate a functional integral over fermions. The subsequent integration over the 1-form e remains Gaussian

and yields a contribution of the form (3.9) to the partition function. The remaining integral over the elements of the holonomy group Ω may prove to be divergent despite the compactness of this group. Indeed, certain eigenvalues λ_q may vanish under certain configurations of the field Ω. Since the expression under the integral sign depends on the negative powers of λ_q, the integral over the field Ω may prove to be divergent. From the physical point of view, these divergences are of great interest. Note that the tendency of eigenvalues λ_q to zero implies that the integral over the 1-form e^a is saturated when the absolute values of this field e^a (or its certain components) tend to infinity. This means that the size of universe tends to infinity (see (2.16)). On the other hand, as will be shown below, the fact that the field components e^a have large values implies that the dynamics of the system becomes quasiclassical. Therefore, from the physical viewpoint, these divergences imply birth of quasiclassical macroscopic space-time.

Concerning the problem under discussion, we note that the presence of Dirac fields in integral (3.1) only strengthens the divergence under the integration over the field e^a. Indeed, after the integration over the fermion field, the integral over the field e^a is rewritten as (cf. (3.6) and (3.7))

$$\mathcal{I} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\, z \, P_n(z) \cdot \exp\left(\frac{i}{2} \lambda z^2\right) , \qquad (3.10)$$

where $P_n(z)$ is a polynomial in z of degree n. For small λ, integral (3.10) is proportional to $|\lambda|^{-(n+1)/2}$.

A similar physical interpretation of divergences under the integration over the field e^a in the continuum quantum B-F-theory in a three-dimensional space-time was given by Witten in [25].

Let us notice another possible type of divergences in a discrete quantum gravity. If the partition function (3.1) was defined for a metric with Lorentzian signature, then the elements of the holonomy group would be the noncompact group $\mathrm{Spin}(3, 1)$. The gauge group (2.14) would also be noncompact, being a direct product of the $\mathcal{V}$ copies of the group $\mathrm{Spin}(3, 1)$. Since both the measure and action in the transfer-matrix are gauge invariant, the functional integral in the transfer-matrix would not be defined at all before the fixation (at least partial) of the gauge. However, the fixation of the gauge in the fundamental transfer-matrix seems to be a so artificial procedure that the theory itself looses its beauty and sense. In our opinion, this means that the fundamental partition function for a discrete theory of gravity can be constructed only on the basis of a metric with Euclidean signature.

In their well-known paper [18], Hartle and Hawking made a hypothesis that the wave function of the universe must be calculated with the use of the functional

integral on the basis of a metric with Euclidean signature. But in the case of the gravity theory the Euclidean action is not positively defined. In our opinion, the arguments for a metric with Euclidean signature provided by the discrete theory of gravity are much more reliable than the arguments given in [18].

IV. HIGH TEMPERATURE EXPANSION

From the beginning let us consider the integral (3.1) in the region of integration variables where

$$|e_{ij}^a| > l_0 \gg l_P \,. \tag{4.1}$$

In this region each item in the sum (2.9) generally is also large since the items in the sum (2.9) are polynomials in the variables e_{ij}^a of powers not less than two. Therefore the whole integral in (3.1) can be estimated quasiclassically or by the stationary phase method. In this region one must use the long wave limit action (1.1), and to perform the stationary phase calculations the integration paths in (3.1) must be deformed so that Lorentzian signature is realized. We see that in the considered model the time arises dynamically in continuum limit. The study of continuum limit of the theory is performed in the subsequent sections.

Now let us consider the integral (3.1) in the region of integration variables where

$$|e_{ij}^a| < l_1 \ll l_P \,. \tag{4.2}$$

In this region each item in the sum (2.9) is small, so that the subintegral quantity in (3.1) (in the case of pure gravity and zero Λ-term) can be written as

$$\exp\left(-I\right) = \prod_A \prod_{i,j,k,l,m} \left(1 + \frac{1}{5 \times 24 \times 2 \times l_P^2} \, \varepsilon_{Aijklm} \operatorname{tr} \gamma^5 \times \right.$$
$$\left. \times \Omega_{Ami} \Omega_{Aij} \Omega_{Ajm} \hat{e}_{Amk} \hat{e}_{Aml} + \dots \right). \tag{4.3}$$

The expansion (4.3) is called here as high-temperature expansion. It is well known that the analogous representation for the $\exp\left(-I\right)$ is true in the lattice Yang–Mills theory in the limit of large coupling constant. From such representation the significant phenomenon of colour confinement follows. Originally the phenomenon of colour confinement has been obtained analytically with the help of high-temperature expansion (with the help of representation of the tipe (4.3)) by Wilson, and then numerous computer simulations confirmed this conclusion. Since the situations concerning high-temperature expansion in both theories are closely analogous, we

make the conclusion that in the region of variables (4.2) also take place colour confinement. Introduce the following notations

$$C = \{a_{i_0}a_{i_1},\, a_{i_1}a_{i_2},\, \dots,\, a_{i_r}a_{i_0}\}$$

is a closed contour or a one dimensional subcomplex with zero boundary;

$$W\left(C\right) = \langle\, \mathrm{tr}\left(\Omega_{i_0 i_1}\Omega_{i_1 i_2}\dots\Omega_{i_r i_0}\right)\,\rangle_1$$

is Wilson loop correlator which in our case is calculated in the theory of pure gravity with zero Λ-term in the region of variables restricted by inequalities (4.2). Let σ_C be a two dimensional subcomplex with boundary $\partial\,\sigma = C$ and $n_C(\sigma)$ be the number of triangles containing in σ_C, and

$$n_C = \min_{\sigma}\{n_C(\sigma)\}\,.$$

Then the simple standard calculations give the following estimation

$$W\left(C\right) \sim \exp\left(-n_C\mu\,\ln l_1^{-1}\right)\,. \tag{4.4}$$

Here μ is a positive number which does not depend on contour C and parameter l_1.

Let us emphasize that in the case of discrete quantum gravity the role of colour gauge group plays the group (2.14). Thus only singlet (i.e. scalar, but not spinor, vector and so on) fields with respect to the group (2.14) have quasiparticle excitations in the region (4.2), i.e. on the early stages of universe development. This conclusion partially justifies the use only scalar fields in numerous works in which the dynamics of early universe is investigated. But in contrast to the Yang–Mills theory in expanding universe the phase transition occurs to deconfinement phase (formally in the region (4.1)). In this phase the dynamics becomes quasiclassical.

Another variant of high temperature expansion, but with the same confined quantum numbers, is presented in [21].

V. THE QUALITATIVE BEHAVIOUR OF ELLIPTIC OPERATORS SPECTRUM ON RANDOM BREATHING LATTICE

Let us now show that the modes of quantized fields in the quasiclassical continuum phase have essentially noncompact packing in momentum space. This important conclusion follows from high-temperature expansion and the most general properties of spectrum of elliptic operators.

We illustrate the effect in Appendix A on the example of the spectrum of one dimensional discrete Laplace operator on random lattice on a cycle. In the cases of 3 and 4 vertexes the problem is solved exactly and we see that in the case when the

total length of the cycle is fixed but the distances between some vertexes tend to zero some of eigenvalues of the operator tend to infinity as an inverse first or second power of the small distances between the corresponding vertexes. This phenomenon of eigenvalues noncompact packing in the case of strongly random lattice I call "spectrum loosening".

Now let us consider the eigenfunction problem for Dirac operator on three-dimensional complex $\mathfrak{S} = \partial\mathfrak{K}$ and observe the spectrum loosening as a consequence of randomness of the lattice. In our case, let index A enumerates tetrahedra. Analogously to the four-dimensional case, $\varepsilon_{Aijkl} = \pm1$ depending on whether the order of vertices $a_{Ai}a_{Aj}a_{Ak}a_{Al}$ defines the positive or negative orientation of the corresponding tetrahedron.

In the three-dimensional case the fermion part of the action (cf. formula (2.9)) can be written as

$$I\{\overline{\psi},\, \psi,\, e\} \equiv I_\psi = \frac{1}{2 \cdot 4 \cdot 6}\sum_A \sum_{i,j,k,l} \varepsilon_{A\,lijk}\varepsilon^{abc}\,\Theta^a_{A\,li}e^b_{A\,lj}e^c_{A\,lk}\,. \tag{5.1}$$

Since the volume of three-dimensional complex (compare with Eq. (2.13))

$$V = \frac{1}{3!} \cdot \frac{1}{4!}\sum_A \sum_{i,j,k,l} \varepsilon_{A\,lijk}\varepsilon^{abc}e^a_{Ali}e^b_{Alj}e^c_{Alk}\,, \tag{5.2}$$

the scalar product in the space of Dirac field modes

$$\langle\psi_P|\psi_Q\rangle = \frac{1}{3!} \cdot \frac{1}{4!}\sum_A \sum_{i,j,k,l} \overline{\psi}_{P\,Al}\psi_{Q\,Al}\,\varepsilon_{A\,lijk}\varepsilon^{abc}e^a_{Ali}e^b_{Alj}e^c_{Alk}\,. \tag{5.3}$$

Here the indices P and Q enumerate the modes of Dirac field. Evidently, in continuum limit

$$\langle\psi_P|\psi_Q\rangle \longrightarrow \int \mathrm{d}^3x\,\overline{\psi}_P(x)\psi_Q(x)\,. \tag{5.4}$$

Let us consider the problem of extremum of quadratic form (5.1) under the condition $\langle\psi|\psi\rangle = 1$. According to the Lagrange method this problem is equivalent to the following one

$$\frac{\delta}{\delta\overline{\psi}_i}\left(I\{\overline{\psi},\, \psi,\, e\} - \epsilon\langle\psi|\psi\rangle\right) = 0\,. \tag{5.5}$$

Let $\{\psi_{Pi}\}$ be a complete orthonormal set of solutions of Eq. (5.5), so that

$$\langle\psi_P|\psi_Q\rangle = \delta_{PQ}\,, \qquad I\{\overline{\psi}_P,\, \psi_P,\, e\} = \epsilon_P\,. \tag{5.6}$$

Using Eqs. (5.5) and (5.6), for any configuration of field

$$\psi_i = \sum_P c_P \psi_{P\,i} \, ,$$

we find:

$$\frac{I\{\overline{\psi}, \, \psi, \, e\}}{\langle \psi | \psi \rangle} = \frac{\sum_P |c_P|^2 \, \epsilon_P}{\sum_P |c_P|^2} \, .$$

From here the following evident estimation follows

$$\left| \frac{I\{\overline{\psi}, \, \psi, \, e\}}{\langle \psi | \psi \rangle} \right| \le |\epsilon_P|_{max} \, . \tag{5.7}$$

In the right hand side of the last inequality the quantity $|\epsilon_P|_{max}$ designates the maximum value among the set of values $\{|\epsilon_P|\}$.

Although complex $\mathfrak{S}$ has the topology of 3-sphere S^3, this circumstance does not play significant role in the present consideration. Therefore, taking into account that $\alpha_q \to \infty$, we shall consider any small part of complex $\mathfrak{S}$ as a three-dimensional complex $\mathfrak{S}'$ embedded in a three-dimensional Euclidean space. To simplify the consideration, we will assume that

$$\Omega_{ij} = 1 \, , \quad \left(e_{ij}^a + e_{jk}^a + \ldots + e_{li}^a \right) = 0 \, . \tag{5.8}$$

Here, the sum in the parentheses is taken over any closed path formed by 1-simplices belonging to complex $\mathfrak{S}'$ and Ω_{ij} is the connection assigned to the 1-simplex $a_i a_j \in \mathfrak{S}'$. Equations (5.8) indicates that the curvature and torsion are equal to zero. Thus, the complex $\mathfrak{S}'$ is in the three-dimensional Euclidean space, e_{ij}^a being the components of the vector in a certain orthogonal basis in this space, and if R_i^a is the radius-vector of vertex a_i, then $e_{ij}^a = R_j^a - R_i^a$.

Let's show that some values of $|\epsilon_P|$ increase without limit on breathing lattice with fixed total volume of subcomplex $\mathfrak{S}'$. Under the "breathing lattice" I understand here the situation when the quantities $\{e_\mathcal{E}\}$ are independent dynamical variables. This means, of course, that the wave function of universe depends also on the set of variables $\{e_\mathcal{E}\}$, and averaging-out over quantum fluctuations includes also averaging-out over the introduced higher values $\{R_i^a\}$

To advance further, let's perform some consideration.

Write out the part of fermion action depending only on spinors associated with neighboring vertexes a_i and a_j belonging to the boundary of 1-simplex $a_i a_j$. Let s_A^3, $A = 1, \ldots, n$ be the complete set of tetrahedrons, each of which contains 1-simplex $a_{Ai} a_{Aj}$. Denote by

$$\mathfrak{v} = \cup_{A=1,\ldots,\,n} s_A^3 \in \mathfrak{S}$$

the minimal three-dimensional subcomplex such that $a_i a_j \notin \partial\mathfrak{v}$, but $a_i \in \partial\mathfrak{v}$ and $a_j \in \partial\mathfrak{v}$. Let $a_k \in \partial\mathfrak{v}$, $k = 1, \ldots, n$ be the rest of vertexes of complex $\mathfrak{v}$ which does not coincide with a_i and a_j and $a_k a_{k+1}$ be a 1-simplex belonging to $\partial\mathfrak{v}$. Thus it is clear that 1-simplex $a_n a_1$ exists and it belong to the boundary $\partial\mathfrak{v}$. For convenience, we assume that index k is defined in (mod n), so that $a_n a_{n+1} = a_n a_1$. It is assumed also that the motion

$$a_1 \to a_2 \to \ldots \to a_n \to a_1$$

appears to be positive (counterclockwise) to an "observer" located at vertex a_j. Introduce the notation

$$S^a_{ij} = \frac{1}{2} \sum_{k=1}^{n} \varepsilon^{abc} e^b_{jk} e^c_{j,k+1} .$$

It is not difficult to show that the part of fermion action depending only on spinors associated with neighboring vertexes a_i and a_j is

$$I_{\psi\, ij} = \frac{i}{12} \overline{\psi}_i \gamma^a S^a_{ij} \psi_j + h.c. . \tag{5.9}$$

Let the configuration $\{\psi_k\}$ be such that $\psi_k = 0$ for $k \neq i$ and $k \neq j$, and

$$\overline{\psi}_i \psi_i \sim \overline{\psi}_j \psi_j \sim 1 . \tag{5.10}$$

In other words, the spinors ψ_k differ from zero only on vertexes a_i and a_j. We assume also that in the vicinity of vertexes a_i and a_j

$$l^2_{kl} < a^2 \tag{5.11}$$

(see formula (2.16)). Then, combining the formulae (5.3), (5.7) and (5.9), we find the following estimation

$$\left| \frac{I\{\overline{\psi}, \psi, e\}}{\langle \psi | \psi \rangle} \right| \sim \frac{1}{a} . \tag{5.12}$$

Thus we see that

$$|\epsilon_P|_{max} \sim a^{-1} \xrightarrow[a \to 0]{} \infty . \tag{5.13}$$

One must emphasize that the total volume (5.2) of the space $\mathfrak{S}$ is maintained constant when the parameter a in (5.11) and (5.13) tends to zero.

Let's pay attention to the well known result that on periodic lattice with the lattice spacing a the estimation (5.13) also is valid. But in this case the limit in (5.13) is impossible if the total volume and the number of vertexes of the lattice are fixed.

Now one needs take into account the fact that the lattice in our case is not only random but it is also breathing since the quantities e_{ij}^a are the dynamical variables. Further we keep in mind the scalar field since the spinor structure does not affect significantly the estimation.

Firstly, we write out the trivial formula for the volume in momentum space occupied by N modes placed in the flat volume V and densely packed in momentum space

$$\Omega = (2\pi)^3 \frac{N}{V}. \tag{5.14}$$

One can easily come to Eq. (5.14) by analysing the discrete Laplace equation on periodic lattice. At first let's consider for simplicity one-dimensional lattice, the vertexes of which are numerated in series by the whole numbers $n = 0, 1, \ldots, N \gg 1$. The complete set of orthonormal eigenfunctions of discrete Laplace operator we take as

$$\psi_k(n) = \frac{1}{\sqrt{N}} e^{ikn}, \qquad \psi_k(0) = \psi_k(N).$$

From here we obtain the quantization conditions for the quasi-momenta

$$k_l = \frac{2\pi l}{N}, \qquad l = 0, \pm 1, \ldots, \pm(N/2).$$

Thus, the maximal difference of the quasi-momentum values is

$$\Delta_{max} k = \frac{2\pi}{N} \Delta_{max} l = 2\pi.$$

Now let's pass to the dimensional quantities by introducing the lattice spacing a. Then $L = Na$ is the length or volume of the space, and $\Delta_{max} p = \Delta_{max} k / a$ is the volume in the momentum space which is occupied by the whole set of modes. Thus we have

$$\Delta_{max} p \equiv \Omega = \frac{\Delta_{max} k}{a} = 2\pi \frac{N}{L}. \tag{5.15}$$

Obviously, the quantity (5.15) is the minimal volume in momentum space which can be occupied by the complete set of momenta of N independent modes of the Laplace operator in concidered problem. Equation (5.14) is obtained now by exponentiation of the right hand side of Eq. (5.15) in third power.

Take into account the fact that in confinement phase all correlators of fundamental fields drop exponentially with space separation. Moreover, the correlators of color fields in space-time representation are proportional to δ-functions. This means

that the fields at nearest regions of space volume are not correlated. It is natural to assume that the same conclusion remains true at initial times in quasiclassical phase. Therefore let us divide a macroscopic volume V with the total number of degrees of freedom (or the number of modes) N into $\mathcal{N}$ subvolumes v_i in each of which n_i degrees of freedom is contained. Thus

$$\sum_{i=1}^{\mathcal{N}} n_i = N \,, \qquad \sum_{i=1}^{\mathcal{N}} v_i = V \,, \tag{5.16}$$

and

$$\omega_i = (2\pi)^3 \frac{n_i}{v_i} \tag{5.17}$$

is the minimal possible volume in momentum space occupied by n_i modes placed in the flat volume v_i. Now instead of the quantity (5.14) one must consider the following quantity

$$\tilde{\Omega} = \frac{(2\pi)^3}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{n_i}{v_i} \,. \tag{5.18}$$

Indeed, the minimum of quantity (5.18) subjected to the constraints (5.16) is equal to (5.14).

Since in the considered theory the volumes v_i are variable quantities, one must introduce the measure on the manifold of volumes $\{v_i\}$. The simplest measure agreeing with fundamental measure (3.2) looks like as follows

$$\mathrm{d}\mu = \frac{(\mathcal{N}-1)!}{V^{\mathcal{N}-1}} \delta\left(V - \sum_{i=1}^{\mathcal{N}} v_i\right) \prod_{i=1}^{\mathcal{N}} \mathrm{d}v_i \,, \quad v_i > 0 \,,$$

$$\int \mathrm{d}\mu = 1 \,. \tag{5.19}$$

To justify the measure (5.19) we give the following arguments:

1) The elementary volumes are given by Eq. (2.13) from which it is seen that the volumes are determined only by 1-forms e_{ij}^a.

2) The variables $\{e_{ij}^a\}$ change independently in integral (3.1).

3) It is important here that the action (2.9) remains almost unchanged under the changes of the variables $\{e_{ij}^a\}$ on a large range. This assertion becomes rigorous in continuum limit if only smooth long-waves are taken into account with wavelengths mich greather than lattice spacing. Indeed, the 1-forms $\omega_{Am\mu}$ and $e_{Am\mu}$ in Eqs. (2.21) and (2.27) do not change under the change of the right-hand sides of these

equations and simultaneous changes of the differentials $\mathrm{d}\,x^{\mu}_{Ami}$. In other words, the quantity e^a_{ij} is the value of differential form $e^a_{\mu}\,\mathrm{d}\,x^{\mu}$ on the vector e^a_{ij}. But the continuum action (1.1) depends on the fields e^a_{μ}, ω^a_{μ}. Thus we see that the changes of the elementary volumes in Eq. (5.19) due to the changes of the variables $\{e^a_{ij}\}$ weakly affect the continuum action. Hence, the wave function remains almost unchanged under the changes of the variables $\{e^a_{ij}\}$ on a large range. Let's designate by

$$\mathcal{D}e \equiv \prod_{\mathcal{E}\in\mathfrak{S}}\prod_a \mathrm{d}\,e^a_{\mathcal{E}} \tag{5.20}$$

the functional measure with the help of which the scalar products of wave functions are calculated. The measure (5.20) can be factorized

$$\mathcal{D}e = \mathcal{D}e_1\,\mathrm{d}\,\mu. \tag{5.21}$$

By definition, the submeasure $\mathcal{D}e_1$ in (5.21) does not depend on the volume variables $\{v_i\}$ and, conversely, the quantity (5.18) does not depend on the variables determining the submeasure $\mathcal{D}e_1$.

Hence, instead of (5.18) the more physically sensible quantity is

$$\langle\tilde{\Omega}\rangle \equiv \int \tilde{\Omega}\,\mathrm{d}\,\mu = (2\pi)^3\frac{\mathcal{N}-1}{V\,\mathcal{N}}\sum_{i=1}^{\mathcal{N}} n_i \int_{v_i\ll V}\frac{\mathrm{d}\,v_i}{v_i} =$$
$$= (2\pi)^3\frac{N}{V}\int_{v_i\ll V}\frac{\mathrm{d}\,v_i}{v_i}. \tag{5.22}$$

The last equality is obtained at taking into account the first constraint of (5.16) and the relation $\mathcal{N}\gg 1$.

The comparison of Eqs. (5.14) and (5.22) shows that taking into account the dynamics of the system leads to the essential expansion of the momentum space volume occupied by quantum field modes. This expansion factor is

$$\varkappa_1 = \int_{v_i\ll V}\frac{\mathrm{d}\,v_i}{v_i} = 3\ln\frac{a_1}{a_0} = 3\ln\xi_0\,. \tag{5.23}$$

Here a_0 is some minimal dimension of the theory and $a_1 \ll V^{1/3}$. It seems that $a_0 \gg l_P$, since only at $|e^a_{ij}| \gg l_P$ the quasiclassical phase can exist (see (4.1)).

But it is not the end of story. From the obtained estimation (5.23) we elaborate a kind of renormalization group describing loosening of mode packing. Let n be the number of steps of renormalization group and

$$\xi_s = \frac{a_{s+1}}{a_s} = \xi \gg 1\,, \quad s = 1,\ldots,n\,, \tag{5.24}$$

and $a_{n+1} = a$ is the radius of universe. Thus $\xi^n = \xi_1 \xi_2 \ldots \xi_n = a/a_0$. For rough estimation let us take

$$n = \frac{1}{\lambda} \ln \frac{a}{a_0} \gg 1, \quad \lambda \gg 1. \tag{5.25}$$

Using Eqs. (5.23)–(5.25) it is easy to see that the expansion factor of momentum space volume occupied by modes after n steps is

$$\varkappa_n = \prod_{s=1}^{n} (3 \ln \xi_s) = (3 \ln \xi)^n = \left(\frac{a}{a_0} \right)^{(\ln 3\lambda)/\lambda}. \tag{5.26}$$

The value of right hand side of Eq. (5.26) can bee very large (many orders) in magnitude. This phenomenon is called here as "spectrum loosening". It seems that the effect of spectrum loosening and translational invariance are compatible only on breathing lattice.

The more careful study of the problem is possible only if we take into account the dynamics of the system.

VI. CONCLUSION

It follows from the presented analysis, that the continuum quantum gravity arising from the discrete quantum gravity (if it exists) possess very unusual properties. For example, let's try to estimate the contribution to cosmological constant due to quantum field fluctuations in the framework of presented here theory. We shall see that the unsolvable problem of a large value of cosmological constant can be solved if the estimation (5.26) is taken into account.

It is well known the estimation for the cosmological constant in the framework of the usual quantum field theory (see, for example [26])

$$\lambda_{eff} \sim G \Lambda^4 \sim l_P^2 \Lambda^4. \tag{6.1}$$

Here G is the Newtonian gravitational constant, l_P is the Planck space-time scale, and Λ is the cutoff parameter in momentum space which usually is restricted by the Planck scale: $\Lambda \sim l_P^{-1}$. Thus the estimation (6.1) becomes as follows

$$\lambda_{eff} \sim l_P^{-2}. \tag{6.2}$$

It should be noted that this estimation is valid i) independently on the size of universe and ii) under the tacit assumption of compact packing of the field modes in momentum space.

But in the elaborated here theory one should correct the estimation (6.2) by the factor $\varkappa_n^{-1} \ll 1$ (see (5.26)) for the reason of noncompact packing of the field modes

in momentum space! Thus, instead of estimation (6.2) now we have the following one

$$\lambda_{eff} \sim \left(\frac{a_0}{a}\right)^{(\ln 3\lambda)/\lambda} l_P^{-2} \lll l_P^{-2} \,. \tag{6.3}$$

So the effective cosmological constant can be made enough small. Thus a possibility of solving the cosmological constant problem arises. This possibility will be elaborated in the subsequent papers.

APPENDIX A

Let us consider the discrete Laplace operator on a one dimensional cycle with 3 vertexes (see fig. 1). The numbers a, b, c are the distances between the vertexes 1 and 2, 2 and 3, 3 and 1, correspondingly. In the vertexes 1, 2 and 3 the real numbers φ_1, φ_2 and φ_3 are defined. Write out the discrete equation for Laplace operator eigenfunctions

$$-(\Delta\varphi)_1 = -\frac{2}{ac}\left(\frac{a\,\varphi_3 + c\,\varphi_2}{a + c} - \varphi_1\right) = \epsilon\,\varphi_1\,,$$

$$-(\Delta\varphi)_2 = -\frac{2}{ab}\left(\frac{b\,\varphi_1 + a\,\varphi_3}{a + b} - \varphi_2\right) = \epsilon\,\varphi_2\,,$$

$$-(\Delta\varphi)_3 = -\frac{2}{bc}\left(\frac{c\,\varphi_2 + b\,\varphi_1}{b + c} - \varphi_3\right) = \epsilon\,\varphi_3\,. \tag{A1}$$

For enough smoothly varying (from vertex to vertex) variables φ_i the system of equations (A1) transforms to the continuum equation $-\Delta\varphi = \epsilon\varphi$. The three eigenvalues of Eq. (A1) are as follows

$$\epsilon_1 = 0\,,$$

$$\epsilon_{2,3} = \frac{a + b + c}{abc}\left[1 \pm \sqrt{1 - \frac{8\,abc}{(a + b)(a + c)(b + c)}}\right]\,. \tag{A2}$$

If $a \to 0$ and $(a + b + c) = \text{const}$, then

$$\epsilon_2 \sim \frac{2(b + c)}{abc} \longrightarrow \infty\,, \qquad \epsilon_3 = \frac{4}{bc}\,. \tag{A3}$$

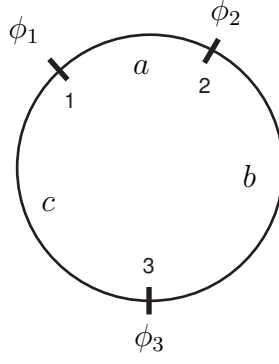FIG. 1:

Consider the same problem for the discrete Laplace operator on a one dimensional cycle with 4 vertexes separated in order by distances a, b, c and d. Then the eigenvalues of the operator satisfy the following equation

$$
\epsilon^4 - 2\epsilon^3 \left(\frac{1}{cd} + \frac{1}{bc} + \frac{1}{ab} + \frac{1}{ad} \right) +
$$

$$
+4\epsilon^2 \left[\frac{1}{bc^2d} + \frac{1}{ab^2c} + \frac{1}{acd^2} + \frac{1}{a^2bd} + \frac{2}{abcd} - \right.
$$

$$
- \frac{1}{c^2(b+c)(c+d)} - \frac{1}{b^2(a+b)(b+c)} -
$$

$$
\left. - \frac{1}{a^2(a+b)(a+d)} - \frac{1}{d^2(a+d)(c+d)} \right] -
$$

$$
-8\epsilon \left[\frac{1}{ab^2c^2d} + \frac{1}{abc^2d^2} + \frac{1}{a^2bcd^2} + \frac{1}{a^2b^2cd} - \right.
$$

$$
- \frac{a+c}{ab^2cd(a+b)(b+c)} - \frac{b+d}{a^2bcd(a+d)(a+b)} -
$$

$$
\left. - \frac{b+d}{abc^2d(b+c)(c+d)} - \frac{a+c}{abcd^2(a+d)(c+d)} \right] = 0 \,. \tag{A4}
$$

Though the exact solution we did not obtain, the approximate solutions of this equation in two interesting here special cases we write out

$$
b = d = l \,, \qquad a \to 0 \,, \qquad c \to 0 \,:
$$

$$
\epsilon_1 = 0 \,, \quad \epsilon_2 \approx \frac{4}{l^2} \,,
$$

$$
\epsilon_3 \approx \frac{4}{la} \to \infty \,, \quad \epsilon_4 \approx \frac{4}{lc} \to \infty \,. \tag{A5}
$$

$$c = d = l\,, \qquad a \to 0\,, \qquad b \to 0\,:$$
$$\epsilon_1 = 0\,, \quad \epsilon_{2,3} \approx \frac{2}{l(a+b)} \to \infty\,,$$
$$\epsilon_4 \approx \frac{2}{ab} - \frac{4}{l(a+b)} \to \infty\,. \tag{A6}$$

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[28] Here, by an almost smooth surface, we mean a piecewise smooth surface consisting of flat four-dimensional simplices, such that the angles between adjacent 4-simplices tend to zero and the sizes of these simplices are commensurable.
[29] We mean here that the partition function is a functional of the values of the dynamic variables at the boundary $\partial\,\mathfrak{K}$.